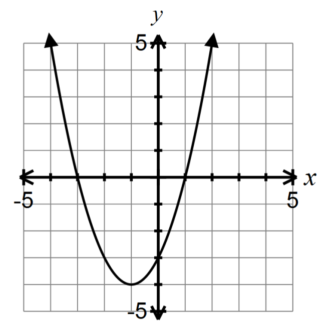
Inequalities Involving Quadratic Functions

Examples: Solve each inequality using the graph of $f(x) = x^2 + 2x - 3$.

Notice that each of these inequalities involves the value of $x^2 + 2x - 3$, which is represented by the *y*-coordinate of the graph. In each case, we are trying to figure out what *x*-values (*x*-coordinates) make the inequality true. When trying to find where $x^2 + 2x - 3 > 0$, we are trying to figure out what *x*-coordinates have a *y*-coordinate that is bigger than zero—in other words, *where is the graph above the x-axis?*

$f(x) = x^2 + 2x - 3$



a) $x^2 + 2x - 3 > 0$

$y > 0$ (above x-axis)

$\boxed{(-\infty, -3) \cup (1, \infty) \text{ or } \{x \mid x < -3 \text{ or } x > 1\}}$

b) $x^2 + 2x - 3 \ge 0$

$y \ge 0$ (on or above x-axis)

$\boxed{(-\infty, -3] \cup [1, \infty) \text{ or } \{x \mid x \le -3 \text{ or } x \ge 1\}}$

c) $x^2 + 2x - 3 < 0$

$y < 0$ (below x-axis)

$\boxed{(-3, 1) \text{ or } \{x \mid -3 < x < 1\}}$

d) $x^2 + 2x - 3 \le 0$

$y \le 0$ (on or below x-axis)

$\boxed{[-3, 1] \text{ or } \{x \mid -3 \le x \le 1\}}$

Solving a Quadratic Inequality in One Variable Graphically:

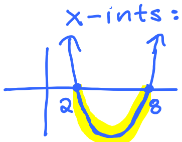
1. Write the inequality in standard form. Replace the inequality sign with an equal sign and solve the equation $ax^2 + bx + c = 0$ by factoring, completing the square, or using the quadratic formula. This gives you the *x*-intercepts of the graph of $y = ax^2 + bx + c$.
2. Graph the related function $y = ax^2 + bx + c$. The graph does not have to be very detailed. A rough sketch of a parabola opening in the correct direction with the correct *x*-intercepts is all you need.
3. The solutions of $ax^2 + bx + c > 0$ are the *x*-values for which the graph is **above** the *x*-axis.
   The solutions of $ax^2 + bx + c \ge 0$ are the *x*-values for which the graph is **on or above** the *x*-axis.
   The solutions of $ax^2 + bx + c < 0$ are the *x*-values for which the graph is **below** the *x*-axis.
   The solutions of $ax^2 + bx + c \le 0$ are the *x*-values for which the graph is **on or below** the *x*-axis.
4. If the inequality involves $\le$ or $\ge$, the *x*-intercepts **are included** in the solution set (use brackets).
   If the inequality involves < or >, the *x*-intercepts **are not included** in the solution set (use parentheses).

Example: Solve each inequality and graph the solution set on a number line.

a) $x^2 - 10x + 16 \le 0$

$(x-8)(x-2) \le 0$ on or below x-axis

x-ints: 8, 2

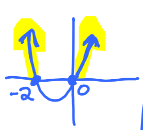


$\boxed{[2, 8] \text{ or } \{x \mid 2 \le x \le 8\}}$

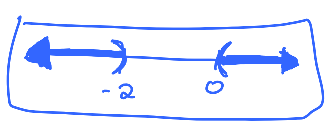
b) $x^2 + 2x > 0$

$x(x+2) > 0$ above x-axis

x-ints: 0, −2



$\boxed{(-\infty, -2) \cup (0, \infty) \text{ or } \{x \mid x < -2 \text{ or } x > 0\}}$



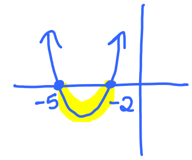
c) $-x^2 - 7x \ge 10$

$0 \ge x^2 + 7x + 10$

$x^2 + 7x + 10 \le 0$

$(x+5)(x+2) \le 0$ on or below x-axis

x-ints: −5, −2



$\boxed{[-5, -2] \text{ or } \{x \mid -5 \le x \le -2\}}$

d) $3x^2+10x-8<0$ below x-axis

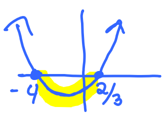
-24	10
-2, 12	10

$(3x^2-2x)+(12x-8)<0$

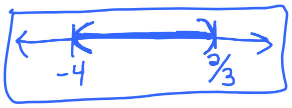
$x(3x-2)+4(3x-2)<0$

$(x+4)(3x-2)<0$

x-ints: $-4,\ \frac{2}{3}$



$\boxed{(-4,\tfrac{2}{3}) \text{ or } \{x \mid -4<x<\tfrac{2}{3}\}}$



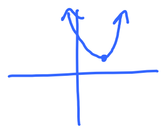
e) $x^2-4x+5\le 0$ on or below x-axis

prime. Use quadratic formula to find x-ints.

$x=\dfrac{4\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}$

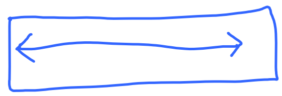
$x=\dfrac{4\pm\sqrt{-16}}{2}=\dfrac{4\pm 4i}{2}=2\pm 2i$

imaginary. No x-ints.



Graph is never on or below x-axis!

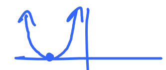
$\boxed{\varnothing}$



f) $x^2+6x+9>0$ above x-axis

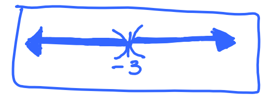
$(x+3)^2>0$

x-int: -3

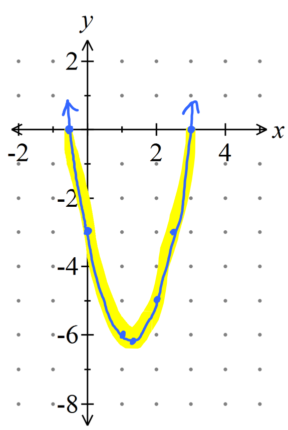


$\boxed{(-\infty,-3)\cup(-3,\infty) \text{ or } \{x \mid x\neq -3\}}$

Graph is on x-axis $(y=0)$ at -3, above x-axis $(y>0)$ everywhere else.



Example: Solve the inequality $2x^2<5x+3$ and graph the solution set.
Method 1: Rearrange the inequality so that 0 is on the right side.



-6	-5
-6, 1	-5

$2x^2-5x-3<0$

$(2x^2-6x)+(x-3)<0$

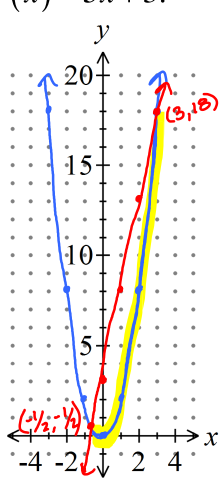
$2x(x-3)+1(x-3)<0$

$(x-3)(2x+1)<0$

x-ints: $3,\ -\frac{1}{2}$

$\boxed{(-\tfrac{1}{2},3) \text{ or } \{x \mid -\tfrac{1}{2}<x<3\}}$

Method 2: Find the intersections of the two functions and determine where $f(x)=2x^2$ is less than $f(x)=5x+3$.



$2x^2=5x+3$

$2x^2-5x-3=0$

$(2x+1)(x-3)=0$

$x=-\frac{1}{2} \qquad x=3$

$\boxed{(-\tfrac{1}{2},3) \text{ or } \{x \mid -\tfrac{1}{2}<x<3\}}$